MATH 2010 Advanced Calculus I
Suggested Solutions for Homework 2

11.4, Q48 Find the volume of a parallelepiped with one of its eight vertices at $A(0,0,0)$, and three adjacent vertices at $B(1,2,0), C(0,-3,2)$ and $D(3,-4,5)$.

Solution Note that

$$\vec{AB} = (1,\ 2,\ 0), \quad \vec{AC} = (0,\ -3,\ 2), \quad \vec{AD} = (3,\ -4,\ 5).$$

Then

$$(\vec{AB} \times \vec{AC}) \cdot \vec{AD} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 3 & -4 & 5 \end{vmatrix} = \begin{vmatrix} -3 & 2 \\ -4 & 5 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 3 & 5 \end{vmatrix} = 5$$

and

$$\text{volume} = |(\vec{AB} \times \vec{AC}) \cdot \vec{AD}| = 5.$$

11.4, Q50 Find a concise 3×3 determinant formula that gives the area of a triangle in the xy-plane having vertices $(a_1,\ a_2), (b_1,\ b_2)$, and $(c_1,\ c_2)$.

Solution

If $A = (a_1,\ a_2), B = (b_1,\ b_2)$ and $C = (c_1,\ c_2)$, then the area of the triangle ABC is $\frac{1}{2}|\vec{AB} \times \vec{AC}|$. Since

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 - a_1 & b_2 - a_2 & 0 \\ c_1 - a_1 & c_2 - a_2 & 0 \end{vmatrix} = \mathbf{k} \begin{vmatrix} b_1 - a_1 & b_2 - a_2 \\ c_1 - a_1 & c_2 - a_2 \end{vmatrix}$$

then

$$\begin{aligned} \frac{1}{2}|\vec{AB} \times \vec{AC}| &= \frac{1}{2}|(b_1 - a_1)(c_2 - a_2) - (b_2 - a_2)(c_1 - a_1)| \\ &= \frac{1}{2}|a_1(b_2 - c_2) + a_2(c_1 - b_1) + (b_1c_2 - c_1b_2)| \\ &= \frac{1}{2}\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}. \end{aligned}$$

11.4, Q54

$$A = (-1,\ 2,\ 3), \quad B = (2,\ 0,\ 1), \quad C = (1,\ -3,\ 2), \quad D = (-2,\ 1,\ -1).$$

Solution Note that

$$\vec{AB} = (3,\ -2,\ -2), \quad \vec{AC} = (2,\ -5,\ -1), \quad \vec{AD} = (-1,\ -1,\ -4).$$

Then

$$(\vec{AB} \times \vec{AC}) \cdot \vec{AD} = \begin{vmatrix} 3 & -2 & -2 \\ 2 & -5 & -1 \\ -1 & -1 & -4 \end{vmatrix} = 53$$

Hence the volume of tetrahedron determined by A, B, C and D is $\frac{53}{6}$.

11.5, Q8 The line through $(2, 4, 5)$ perpendicular to the plan $3x + 7y - 5z = 21$.

Solution

Note that $\mathbf{n} = 3\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}$ is normal the plane $3x + 7y - 5z = 21$. Then the parametric equations for the lines through $(2, 4, 5)$ parallel to $\mathbf{n}$ is

$$x = 2 + 3t, \quad y = 4 + 7t, \quad z = 5 - 5t.$$

11.5, Q10 The line through (2, 3, 0) perpendicular to the vectors $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

Solution

The direction is parallel to $\mathbf{u} \times \mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$$

The line is

$$x = 2 - 2t, \quad y = 3 + 4t, \quad z = -2t.$$

11.5, Q22 The plane through $(1, -1, 3)$ parallel to the plane $3x + y + z = 7$,

Solution

The vector $\mathbf{n} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ is normal to the planes which parallel to the plane $3x + y + z = 7$. Therefore the plane through $(1, -1, 3)$ and normal to $\mathbf{n} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ is given by

$$3(x - 1) + 1(y + 1) + 1(z - 3) = 0 \Rightarrow 3x + y + z = 5.$$

11.5, Q28 Find the intersection of lines

$$x = t, \quad y = -t + 2, \quad z = t + 1$$

$$x = 2s + 2, \quad y = s + 3, \quad z = 5s + 6$$

and then find the plane determined by these lines.

Solution

The intersection lies in both lines, then we have

$$x = t = 2s + 2, \quad y = -t + 2 = s + 3$$

which derives $s = -1$ and $t = 0$. Hence the intersection is $(0, 2, 1)$.

Let $\mathbf{n_1} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{n_2} = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ are directions of above two lines. A vector normal to the plane determined by these lines is

$$\mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix} = -6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}.$$

Hence the plane is

$$-6(x - 0) - 3(y - 2) + 3(z - 1) = 0 \quad \Rightarrow -6x - 3y + 3z = -3.$$

11.5, Q36 Find the distance from the point $\mathbf{P}(2, 1, -1)$ to the line

$$x = 2t, \quad y = 1 + 2t, \quad z = 2t.$$

Solution

We see from the equation that the line pass through the point $\mathbf{O}(0, 1, 0)$ and parallel to $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. Then

$$\vec{OP} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix} = 2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}.$$

and the distance is

$$d = \frac{|\vec{OP} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{2^2 + (-6)^2 + 4^2}}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{\sqrt{14}}{\sqrt{3}}.$$

11.5, Q40 Find the distance from the point $\mathbf{P}(0, 0, 0)$ to the plane

$$3x + 2y + 6z = 6.$$

Solution

It is clear that $\mathbf{O}(2, 0, 0)$ is on the plane and $\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ is normal to the plane. Then

$$\vec{OP} = (-2, 0, 0)$$

and the distance is

$$d = \left|\vec{OP} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-6}{\sqrt{3^2 + 2^2 + 6^2}}\right| = \frac{6}{7}.$$

11.5, Q72 How can you tell when two planes $A_1x + B_1y + C_1z = D_1$ and $A_2x + B_2y + C_2z = D_2$ are parallel? Perpendicular? Give reasons for your answer.

Solution

The planes are parallel when either vector $A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}$ or $A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k}$ is a multiple of the other or when

$$(A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}) \times (A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k}) = 0.$$

The planes are perpendicular when their normals are perpendicular, or

$$(A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}) \cdot (A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k}) = 0.$$